Low-Rank Approximation of MRF Energy by means of the TT-Format

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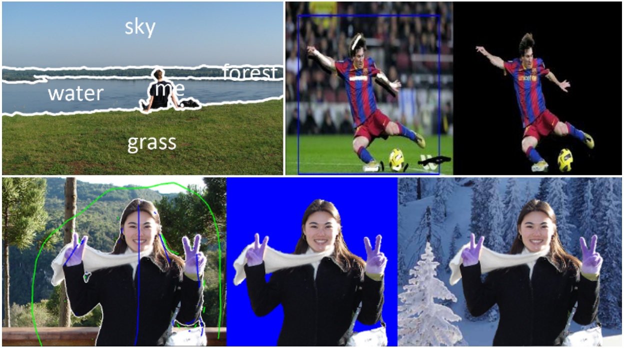
Semantic Image Segmentation



Goal: assign a label $t_i \in \Lambda$ to each pixel of the image.

Problem: for an $M \times N$ image there are $|\Lambda|^{MN}$ possible labellings. Which one is the best?

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- We will use Markov random fields (MRFs) to define the probabilistic model $p(T|X, W)$.

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- Define some positive functions $\Psi_c(T_c; X, W)$ (called MRF factors) on the cliques of the graph $\mathcal{G}$.
- The model is then defined as follows:
$$p(T|X, W) = \frac{1}{Z(X, W)} \prod_{c \in \mathcal{C}} \Psi_c(T_c; X, W),$$
where $Z(X, W)$ is the normalization constant.

Markov Random Fields cont'd

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- How to choose the factors $\Psi_c(T_c; X, W)$?

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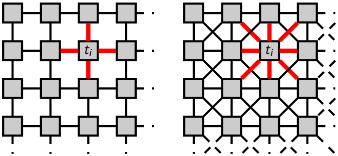

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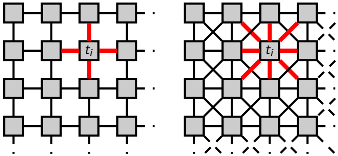

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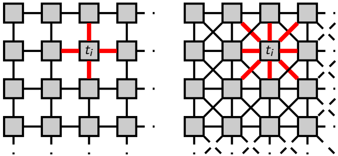

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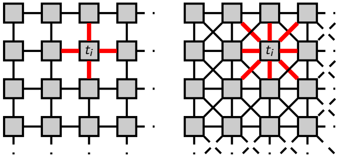

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  - unary factors $\Psi_i(t_i)$: how likely it is that the i-th pixel is labelled as t_i;
  - pairwise factors $\Psi_{ij}(t_i, t_j)$: how likely it is that the i-th and j-th pixels are *simultaneously* labelled as t_i and t_j.

MAP-Inference

The MAP-inference problem now corresponds to the following problem:

$$\max_T p(T|X, W) = \max_T \frac{1}{Z(X, W)} \prod_{c \in \mathcal{C}} \Psi_c(T_c; X, W).$$

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So, to simplify notation, we won't explicitly write X, W any more:

$$\max_T p(T) = \max_T \frac{1}{Z} \prod_{c \in \mathcal{C}} \Psi_c(T_c)$$

Tensor-Train Framework

- An n-dimensional tensor $\boldsymbol{A}$ is said to be represented in the TT-format if its elements can be expressed as the following matrix product:

$$\boldsymbol{A}(x_1, \ldots, x_n) = \underbrace{G_1[x_1]}_{\mathrm{r}_0 \times \mathrm{r}_1} \underbrace{G_2[x_2]}_{\mathrm{r}_1 \times \mathrm{r}_2} \ldots \underbrace{G_n[x_n]}_{\mathrm{r}_{n-1} \times \mathrm{r}_n} .$$

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  - TT-SVD: finds an exact TT-representation for a tensor but suitable only for low dimensionality n.
  - AMEn: builds a TT-approximation of a tensor by using only a small fraction of its elements; suitable for high dimensionality n but doesn't have strong theoretical guarantees.

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The main problem of our interest is the MAP-inference problem:

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So, the MAP-inference is equivalent to energy minimization:

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AMEn-algorithm? Possible, but there is also a much better way!

The Idea of the Algorithm

- Let's try to take into account the structure of the energy tensor $\boldsymbol{E}$.
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- How to find the TT-decomposition for each tensor $\boldsymbol{\Theta}_\ell$?

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    - To introduce x_3, x_5, we need to define the missing cores $G_3[x_3]$, $G_5[x_5]$.

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    - Suppose that after TT-SVD we have:

$$\boldsymbol{\Theta}_\ell(x_1, x_2, x_4) = G_1[x_1] G_2[x_2] G_4[x_4].$$

    - To introduce x_3, x_5, we need to define the missing cores $G_3[x_3]$, $G_5[x_5]$.
    - Define them as identity matrices:

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Converting Potentials into the TT-Format

- As opposed to the energy $\boldsymbol{E}(\boldsymbol{x})$, each potential $\boldsymbol{\Theta}_\ell(\boldsymbol{x}^\ell)$ depends only on part of the all variables and is usually of low dimensionality.
- To compute the TT-decomposition of the tensor $\boldsymbol{\Theta}_\ell(\boldsymbol{x}^\ell)$, we can use the TT-SVD algorithm.
- All that remains is to add the inessential variables $\boldsymbol{x} \setminus \boldsymbol{x}^\ell$ to $\boldsymbol{\Theta}_\ell(\boldsymbol{x}^\ell)$ so as to make it n-dimensional.
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  - The maximal TT-rank hasn't increased!

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Theorem

The maximal TT-rank of the tensor $\boldsymbol{E}$ constructed by the algorithm is polynomially bounded:

$$\mathrm{r}(\boldsymbol{E}) \leq d^{\frac{p}{2}} m,$$

where

- *d is the number of values that each variable can take;*
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Consider $d = 2$, $p = 2$. Then $\mathrm{r}(\boldsymbol{E}) \le 2m$ (linear dependence on m).

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- However, for some of these potentials we can explicitly construct the TT-representation, i.e. we can derive analytical formulas for the corresponding TT-cores.
- Such TT-representations will be of low TT-rank!

Sparse Potential

- Consider a so-called sparse potential:

$$\boldsymbol{\Theta}_\ell(x_{i_1}, \dots, x_{i_w}) = [x_{i_1} = \beta_1] \dots [x_{i_w} = \beta_w].$$

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- In this case each TT-core is simply a number (1-by-1 matrix) for every concrete value of x_{i_v}. Hence, the maximal TT-rank equals 1.
- A more general sparse potential which differs from zero on $s > 1$ configurations can be obtained as a sum of several potentials of the above type. Thus, the TT-rank of a general sparse potential is bounded above by s.

Area Potential

- Consider the potential

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where $x_i \in \{0, 1\}$ and $a \in \mathbb{Z}_+$.

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where $S_a = \underbrace{\left[\begin{array}{c|c} O & I_a \\ \hline O & O \end{array} \right]}_{(a+1)\times(a+1)}.$

Area Potential cont'd

Key property of S_a: $[\overbrace{\underbrace{0\dots0}_{k}\;1\dots1}^{a+1}]S_a = [\overbrace{\underbrace{0\dots0}_{k+1}\;1\dots1}^{a+1}]$.

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Consider, e.g., that $a = 3$. In this case

$$S_a = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

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- and so on.

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1. Convert the energy into the TT-format;
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Problem	**Variables**	**Labels**	**TRW-S**	**TT**	**Time** (sec)
gm6	320	3	45.03	43.11	637
gm29	212	3	56.81	56.21	224
gm66	198	3	75.19	74.92	172
gm105	237	3	67.81	67.71	230
gm32	100	7	150.50	289.29	257
gm70	122	7	121.78	163.60	399
gm85	143	7	168.30	228.40	1 912
gm192	99	7	114.51	174.78	180

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Conclusion

- We have proposed an algorithm that converts MRF energy into the TT-format exactly.
- We have derived an upper bound on the TT-ranks of the energy tensor constructed by the proposed algorithm.
- We have demonstrated how the obtained TT-representation of MRF energy can be used for solving the important problem of the MAP-inference arising in probabilistic graphical models.
- To improve the method, we need a better algorithm for finding the minimal element in a tensor represented in the TT-format.

Thank you!